

# VISCOSITY SOLUTIONS TO COMPLEX HESSIAN EQUATIONS

LU HOANG CHINH

**ABSTRACT.** We study viscosity solutions to complex hessian equations. In the local case, we consider  $\Omega$  a bounded domain in  $\mathbb{C}^n$ ,  $\beta$  the standard Kähler form in  $\mathbb{C}^n$  and  $1 \leq m \leq n$ . Under some suitable conditions on  $F, g$ , we prove that the equation  $(dd^c\varphi)^m \wedge \beta^{n-m} = F(x, \varphi)\beta^n$ ,  $\varphi = g$  on  $\partial\Omega$  admits a unique viscosity solution modulo the existence of subsolution and supersolution. If moreover, the datum are Hölder continuous then so is the solution. In the global case, let  $(X, \omega)$  be a compact hermitian homogeneous manifold where  $\omega$  is an invariant hermitian metric (not necessarily Kähler). We prove that the equation  $(\omega + dd^c\varphi)^m \wedge \omega^{n-m} = F(x, \varphi)\omega^n$  has a unique viscosity solution under some natural conditions on  $F$ .

## 1. INTRODUCTION

The complex Hessian equation has been studied intensively in recent years. In [Li04], Li solved Dirichlet problems for complex Hessian equations in  $m$ -pseudoconvex domains with smooth right-hand-side and smooth boundary data by using the continuity method. In [Bl05], Blocki considered degenerate complex Hessian equations in  $\mathbb{C}^n$  and developed a first step of a potential theory for this equation. Recently, Sadullaev and Abdullaev also studied capacities and polar sets for  $m$ -subharmonic functions [SA12].

Hou [Hou09] and Jbilou [Jb10] began the program of solving the non degenerate complex Hessian equation on compact Kähler manifolds four years ago. They independently solved this equation when the underlined manifold has non negative holomorphic bisectional curvatures. The curvature assumption served as a technical point in an a priori  $\mathcal{C}^2$  estimates and people wanted to remove it. Later on, Hou-Ma-Wu [HMW10] provided an important  $\mathcal{C}^2$  estimate without this hypothesis. Using this estimate and a blowing-up analysis, Dinew and Kolodziej recently solved the equation in full generality [DK12].

Degenerate complex Hessian equations on compact Kähler manifold were considered in [DK11] and [Chi12]. This approach is global in nature since it relies on some difficult integrations by parts.

The study of real Hessian equations is a classical subject which has been developed previously in many papers, for example [CNS85, CW01, ITW04, Kry95, Lab02, Tr95, TW99, Ur01, W09].

The viscosity method introduced in [Lio83] (see also [CIL92] for a survey) is purely local and very efficient to study weak solutions to nonlinear elliptic partial differential equations. In [EGZ11] the authors used this method to study degenerate complex Monge-Ampère equation on compact Kähler manifolds. In the local context, using this approach Wang [YW10] considered the Dirichlet problem for complex Monge-Ampère equations where the right-hand-side also depends on the solution.

From recent developments on viscosity method applied to complex Monge-Ampère equations it is natural to develop such a treatment for complex Hessian equations.

---

*Date:* October 24, 2012.

It is the main purpose of this paper, precisely we consider the following complex Hessian equation:

$$(1) \quad -(dd^c \varphi)^m \wedge \beta^{n-m} + F(x, \varphi) \beta^n = 0,$$

with boundary value  $\varphi = g$  on  $\partial\Omega$ , where  $1 \leq m \leq n$ ,  $\Omega$  is a bounded domain in  $\mathbb{C}^n$ ,  $\beta$  is the standard Kähler form in  $\mathbb{C}^n$ ,

$$(2) \quad g \in \mathcal{C}(\partial\Omega), \text{ and}$$

$$(3) \quad F(x, t) \text{ is a continuous function on } \Omega \times \mathbb{R} \rightarrow \mathbb{R}^+ \\ \text{which is non-decreasing in the second variable.}$$

We say that  $F^{1/m}$  is  $\gamma$ -Hölder continuous uniformly in  $t$  if

$$(4) \quad \sup_{|t| \leq M} \sup_{x \neq y \in \bar{\Omega}} \frac{|F^{1/m}(x, t) - F^{1/m}(y, t)|}{|x - y|^\gamma} < +\infty, \forall M > 0.$$

Equation (1) is non linear degenerate second order elliptic in the viscosity sense (see [CIL92]). So, we can use the concepts of subsolutions and supersolutions.

The main results are the followings:

**Theorem A.** *Let  $g, F$  be functions satisfying (2) and (3) respectively. Assume that there exist a bounded subsolution  $u$  and a bounded supersolution  $v$  to (1) such that  $u_* = v^* = g$  on  $\partial\Omega$ . Then there exists a unique viscosity solution to (1) with boundary value  $g$ . It is also the unique potential solution.*

**Theorem B.** *With the same assumption as in Theorem A, assume moreover that  $u, v$  are  $\gamma$ -Hölder continuous in  $\Omega$  and  $F$  satisfies (4). Then the unique solution of (1) with boundary value  $g$  is also  $\gamma$ -Hölder continuous in  $\Omega$ .*

In Theorem A and Theorem B, to solve the equation we need to find a subsolution and a supersolution. When the domain  $\Omega$  is strictly pseudoconvex these existences are guaranteed.

**Corollary C.** *Assume that  $g \in \mathcal{C}(\partial\Omega)$  and  $F$  satisfies (3) and (4). If  $\Omega$  is strongly pseudoconvex then (1) has a unique viscosity solution with boundary value  $g$ . If, moreover,  $g$  is  $(2\gamma)$ -Hölder continuous and  $F$  satisfies (4) with  $0 < \gamma \leq 1$  then the unique solution is  $\gamma$ -Hölder continuous.*

**Remark.** When  $m = n$  we recover the results in [YW10].

We also study viscosity solutions on compact homogeneous Hermitian manifolds. We assume that  $X$  is a Hermitian manifold with a hermitian metric  $\omega$  such that the following conditions are verified:

- (H1)  $X = G/H$  where  $G$  is a connected Lie group and  $H$  is a closed subgroup.
- (H2) there exists a compact subgroup  $K \subset G$  which acts transitively on  $X$ .
- (H3)  $\omega$  is invariant by  $K$ .

**Theorem D.** *Assume that  $(X, \omega)$  satisfies (H1), (H2) and (H3). Let  $F(x, t)$  be a continuous function which is increasing in  $t$  and assume that there exists  $t_0, t_1 \in \mathbb{R}$  such that*

$$(5) \quad F(x, t_0) \leq 1 \leq F(x, t_1), \forall x \in X.$$

*Then there exists a unique viscosity solution to*

$$-(\omega + dd^c \varphi)^m \wedge \omega^{n-m} + F(x, \varphi) \omega^n = 0.$$

**Remark.** Our proof here does not use a priori  $\mathcal{C}^2$  estimates in contrast with a similar result in [Chi12] where we use potential method [Chi12] and existence result of [DK12] which relies on  $\mathcal{C}^2$  estimate of Hou-Ma-Wu [HMW10]. Moreover, we do not assume that  $\omega$  is closed. An example of compact Hermitian manifold satisfying (H1), (H2), (H3) which is not Kähler was given to us by Professor Karl Oeljeklaus to whom we are indebted (example 6.2).

**Acknowledgement.** The author would like to express his deep gratitude to Professor Ahmed Zeriahi and Professor Vincent Guedj for useful discussions and constant encouragement. The author also would like to thank Professor Philippe Eyssidieux and Professor Dan Popovici for valuable discussions. The author is indebted to Professor Karl Oeljeklaus for giving an important example and references for it and also for very useful discussions.

## 2. PRELIMINARIES

In this section,  $\Omega$  is a bounded domain and  $\beta$  is the standard Kähler form in  $\mathbb{C}^n$ . We introduce the notion and basic properties of  $m$ -subharmonic functions in the local context and one of  $(\omega, m)$ -subharmonic functions on compact Kähler manifolds.

**2.1. Elementary symmetric functions.** We begin by a brief review of elementary symmetric functions (see [Bl05], [CW01], [Ga59]). We use the notations in [Bl05].

Let  $1 \leq k \leq n$  be natural numbers. The elementary symmetric function of order  $k$  is defined by

$$S_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}, \quad \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n.$$

Let  $\Gamma_k$  denote the closure of the connected component of  $\{S_k(\lambda) > 0\}$  containing  $(1, \dots, 1)$ . It is easy to show that

$$\Gamma_k = \{\lambda \in \mathbb{R}^n / S_k(\lambda_1 + t, \dots, \lambda_n + t) \geq 0, \forall t \geq 0\}.$$

and hence

$$\Gamma_k = \{\lambda \in \mathbb{R}^n / S_j(\lambda) \geq 0, \forall 1 \leq j \leq k\}.$$

We have an obvious inclusion  $\Gamma_n \subset \dots \subset \Gamma_1$ .

The set  $\Gamma_k$  is a convex cone in  $\mathbb{R}^n$  and  $S_k^{1/k}$  is concave on  $\Gamma_k$  [Ga59].

Let  $\mathcal{H}$  denote the vector space (over  $\mathbb{R}$ ) of complex hermitian matrices of dimension  $n \times n$ . For  $A \in \mathcal{H}$  we set

$$\tilde{S}_k(A) = S_k(\lambda(A)),$$

where  $\lambda(A) \in \mathbb{R}^n$  are the vector of eigenvalues of  $A$ . The function  $\tilde{S}_k$  can also be defined as the sum of all principal minors of order  $k$ ,

$$\tilde{S}_k(A) = \sum_{|I|=k} A_{II}.$$

From the latter we see that  $\tilde{S}_k$  is a homogeneous polynomial of order  $k$  on  $\mathcal{H}$  which is hyperbolic with respect to the identity matrix  $I$  (that is for every  $A \in \tilde{S}$  the equation  $\tilde{S}_k(A + tI) = 0$  has  $n$  real roots; see [Ga59]). As in [Ga59] (see also [Bl05]), the cone

$$\tilde{\Gamma}_k := \{A \in \mathcal{H} / \tilde{S}_k(A + tI) \geq 0, \forall t \geq 0\} = \{A \in \mathcal{H} / \lambda(A) \in \Gamma_k\}$$

is convex and the function  $\tilde{S}_k^{1/k}$  is concave on  $\tilde{\Gamma}_k$ .

**2.2.  $m$ -subharmonic functions and the Hessian operator.** We associate real  $(1,1)$ -forms  $\alpha$  in  $\mathbb{C}^n$  with hermitian matrices  $[a_{j\bar{k}}]$  by

$$\alpha = \frac{i}{\pi} \sum_{j,k} a_{j\bar{k}} dz_j \wedge d\bar{z}_k.$$

Then the canonical Kähler form  $\beta$  is associated with the identity matrix  $I$ . It is easy to see that

$$\binom{n}{k} \alpha^k \wedge \beta^{n-k} = \tilde{S}_k(A) \beta^n.$$

**Definition 2.1.** Let  $\alpha$  be a real  $(1,1)$ -form on  $\Omega$ . We say that  $\alpha$  is  $m$ -positive at a given point  $P \in \Omega$  if at this point we have

$$\alpha^j \wedge \beta^{n-j} \geq 0, \quad \forall j = 1, \dots, m.$$

$\alpha$  is called  $m$ -positive if it is  $m$ -positive at any point of  $\Omega$ . If there is no confusion we also denote by  $\tilde{\Gamma}_m$  the set of  $m$ -positive  $(1,1)$ -forms.

Let  $T$  be a current of bidegree  $(n-k, n-k)$  with  $k \leq m$ . Then  $T$  is called  $m$ -positive if

$$\alpha_1 \wedge \dots \wedge \alpha_k \wedge T \geq 0,$$

for all  $m$ -positive  $(1,1)$ -forms  $\alpha_1, \dots, \alpha_k$ .

**Definition 2.2.** A function  $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$  is called  $m$ -subharmonic if it is subharmonic and

$$dd^c u \wedge \alpha_1 \wedge \dots \wedge \alpha_{m-1} \wedge \beta^{n-m} \geq 0,$$

for every  $m$ -positive  $(1,1)$ -forms  $\alpha_1, \dots, \alpha_{m-1}$ . The class of all  $m$ -subharmonic functions in  $\Omega$  will be denoted by  $\mathcal{P}_m(\Omega)$ .

We resume basic properties of  $m$ -subharmonic functions in the followings:

**Proposition 2.3.** [Bl05] (i) If  $u$  is  $\mathcal{C}^2$  smooth then  $u$  is  $m$ -subharmonic if and only if the form  $dd^c u$  is  $m$ -positive at every point in  $\Omega$ .

(ii) If  $u, v \in \mathcal{P}_m(\Omega)$  then  $\lambda u + \mu v \in \mathcal{P}_m(\Omega)$ ,  $\forall \lambda, \mu > 0$ .

(iii) If  $u$  is  $m$ -subharmonic in  $\Omega$  then the standard regularization  $u \star \chi_\epsilon$  are also  $m$ -subharmonic in  $\Omega_\epsilon := \{x \in \Omega : d(x, \partial\Omega) > \epsilon\}$ .

(iv) If  $(u_l) \subset \mathcal{P}_m(\Omega)$  is locally uniformly bounded from above then  $(\sup u_l)^* \in \mathcal{P}_m(\Omega)$ , where  $v^*$  is the upper semi continuous regularization of  $v$ .

(v)  $PSH = \mathcal{P}_n \subset \dots \subset \mathcal{P}_1 = SH$ .

(vi) Let  $\emptyset \neq U \subset \Omega$  be a proper open subset such that  $\partial U \cap \Omega$  is relatively compact in  $\Omega$ . If  $u \in \mathcal{P}_m(\Omega)$ ,  $v \in \mathcal{P}_m(U)$  and  $\limsup_{x \rightarrow y} v(x) \leq u(y)$  for each  $y \in \partial U \cap \Omega$  then the function  $w$ , defined by

$$w = \begin{cases} u & \text{on } \Omega \setminus U \\ \max(u, v) & \text{on } U \end{cases},$$

is  $m$ -subharmonic in  $\Omega$ .

For locally bounded  $m$ -subharmonic functions  $u_1, \dots, u_p$  ( $p \leq m$ ) and a closed  $m$ -positive current  $T$  we can inductively define a closed  $m$ -positive current  $dd^c u_1 \wedge \dots \wedge dd^c u_p \wedge T$  (following Bedford Taylor [BT76]).

**Lemma 2.4.** Let  $u_1, \dots, u_k$  ( $k \leq m$ ) be locally bounded  $m$ -subharmonic functions in  $\Omega$  and let  $T$  be a closed  $m$ -positive current of bidegree  $(n-p, n-p)$  ( $p \geq k$ ). Then we can define inductively a closed  $m$ -positive current

$$dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge T,$$

and the product is symmetric, i.e.

$$dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_p \wedge T = dd^c u_{\sigma(1)} \wedge dd^c u_{\sigma(2)} \wedge \dots \wedge dd^c u_{\sigma(k)} \wedge T,$$

for every permutation  $\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ .

In particular, the Hessian measure of  $u \in \mathcal{P}_m(\Omega) \cap L_{loc}^\infty$  is well defined as

$$H_m(u) = (dd^c u)^m \wedge \beta^{n-m}.$$

**2.3.  $(\omega, m)$ -subharmonic functions.** In this section,  $(X, \omega)$  is a compact Kähler manifold and  $U \subset X$  is an open subset contained in a local chart.

**Definition 2.5.** A function  $u \in L^1(U)$  is called weakly  $\omega$ -subharmonic if

$$dd^c u \wedge \omega^{n-1} \geq 0,$$

in the weak sense of currents.

Thanks to Littman [Lit63] we have the following approximation properties.

**Proposition 2.6.** *Let  $u$  be a weakly  $\omega$ -subharmonic function in  $U$ . Then there exists a one parameter family of functions  $u_h$  with the following properties: For every compact subset  $U' \subset U$*

- a)  $u_h$  is smooth in  $U'$  for  $h$  sufficiently large,*
- b)  $dd^c u_h \wedge \omega^{n-1} \geq 0$  in  $U'$ ,*
- c)  $u_h$  is non increasing with increasing  $h$ , and  $\lim_{h \rightarrow \infty} u_h(x) = u(x)$  almost every where in  $U'$ ,*
- d)  $u_h$  is given explicitly as  $u_h(y) = \int_U K_h(x, y) u(x) dx$ , where  $K_h$  is a smooth non negative function and  $\int_U K_h(x, y) dy \rightarrow 1$ , uniformly in  $x \in U'$ .*

**Definition 2.7.** A function  $u$  is called  $\omega$ -subharmonic if it is weakly  $\omega$ -subharmonic and for every  $U' \Subset U$ ,  $\lim_{h \rightarrow \infty} u_h(x) = u(x), \forall x \in U'$ , where  $u_h$  is constructed as in Proposition 2.6.

**Remark 2.8.** Any continuous weakly  $\omega$ -subharmonic function is  $\omega$ -subharmonic.

If  $(u_j)$  is a sequence of continuous  $\omega$ -subharmonic functions decreasing to  $u \not\equiv -\infty$  then  $u$  is  $\omega$ -subharmonic.

If  $u$  is weakly  $\omega$ -subharmonic then the pointwise limit of  $(u_h)$  is a  $\omega$ -subharmonic function.

Let  $(u_j)$  be a sequence of  $\omega$ -subharmonic functions and  $(u_j)$  is uniformly bounded from above. Then  $u := (\limsup_j u_j)^*$  is  $\omega$ -subharmonic. Where for a function  $v$ ,  $v^*$  denotes the upper semicontinuous regularization of  $v$ .

**Definition 2.9.** Let  $\alpha$  be a real  $(1, 1)$ -form on  $X$ . We say that  $\alpha$  is  $(\omega, m)$ -positive at a given point  $P \in X$  if at this point we have

$$\alpha^k \wedge \omega^{n-k} \geq 0, \quad \forall k = 1, \dots, m.$$

We say that  $\alpha$  is  $(\omega, m)$ -positive if it is  $(\omega, m)$ -positive at any point of  $X$ .

**Remark 2.10.** Locally at  $P \in X$  with local coordinates  $z_1, \dots, z_n$ , we have

$$\alpha = \frac{i}{\pi} \sum_{j,k} \alpha_{j\bar{k}} dz_j \wedge d\bar{z}_k,$$

and

$$\omega = \frac{i}{\pi} \sum_{j,k} g_{j\bar{k}} dz_j \wedge d\bar{z}_k.$$

Then  $\alpha$  is  $(\omega, m)$ -positive at  $P$  if and only if the eigenvalues  $\lambda(g^{-1}\alpha) = (\lambda_1, \dots, \lambda_n)$  of the matrix  $\alpha_{j\bar{k}}(P)$  with respect to the matrix  $g_{j\bar{k}}(P)$  is in  $\Gamma_m$ . These eigenvalues are independent of any choice of local coordinates.

Following Blocki [Bl05] we can define  $(\omega, m)$ -subharmonicity for (non smooth) functions.

**Definition 2.11.** A function  $\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\}$  is called  $(\omega, m)$ -subharmonic if the following conditions hold:

- (i) in any local chart  $\Omega$ , given  $\rho$  a local potential of  $\omega$  and set  $u := \rho + \varphi$ , then  $u$  is  $\omega$ -subharmonic,
- (ii) for every smooth  $(\omega, m)$ -positive forms  $\beta_1, \dots, \beta_{m-1}$  we have, in the weak sense of distributions,

$$(\omega + dd^c \varphi) \wedge \beta_1 \wedge \dots \wedge \beta_{m-1} \wedge \omega^{n-m} \geq 0.$$

Let  $SH_m(X, \omega)$  be the set of all  $(\omega, m)$ -subharmonic functions on  $X$ . Observe that, by definition, any  $\varphi \in SH_m(X, \omega)$  is upper semicontinuous.

The following properties of  $(\omega, m)$ -subharmonic functions are easy to show.

**Proposition 2.12.** (i) If  $\varphi \in \mathcal{C}^2(X)$  then  $\varphi$  is  $(\omega, m)$ -subharmonic if the form  $(\omega + dd^c \varphi)$  is  $(\omega, m)$ -positive, or equivalently

$$(\omega + dd^c \varphi) \wedge (\omega + dd^c u_1) \wedge \dots \wedge (\omega + dd^c u_{m-1}) \wedge \omega^{n-m} \geq 0,$$

for all  $\mathcal{C}^2$   $(\omega, m)$ -subharmonic functions  $u_1, \dots, u_{m-1}$ .

(ii) If  $\varphi, \psi \in SH_m(X, \omega)$  then  $\max(\varphi, \psi) \in SH_m(X, \omega)$ .

(ii) If  $\varphi, \psi \in SH_m(X, \omega)$  and  $\lambda \in [0, 1]$  then  $\lambda\varphi + (1 - \lambda)\psi \in SH_m(X, \omega)$ .

(iii) If  $(\varphi_j) \subset SH_m(X, \omega)$  is uniformly bounded from above then

$$(\limsup_j \varphi_j)^* \in SH_m(X, \omega).$$

### 3. VISCOSITY SOLUTIONS VS POTENTIAL SOLUTIONS

In this section we introduce the notion of viscosity (sub, super) solutions to degenerate complex Hessian equations and systematically compare them with potential ones. We prove an important comparison principle which is the key point in the proof of our main results. The idea of our proof is taken from [EGZ11], [YW10, CC95], [CIL92].

**Definition 3.1.** Let  $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$  be a function. Let  $\varphi$  be a  $\mathcal{C}^2$  function in a neighborhood of  $x_0 \in \Omega$ . We say that  $\varphi$  touches  $u$  from above (resp. below) at  $x_0$  if  $\varphi(x_0) = u(x_0)$  and  $\varphi(x) \geq u(x)$  (resp.  $\varphi(x) \leq u(x)$ ) for every  $x$  in a neighborhood of  $x_0$ .

**Definition 3.2.** An upper semicontinuous function  $\varphi : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$  is a viscosity subsolution to

$$(6) \quad -(dd^c \varphi)^m \wedge \beta^{n-m} + F(x, \varphi) \beta^n = 0$$

if  $\varphi \not\equiv -\infty$  and for any  $x_0 \in \Omega$  and any  $\mathcal{C}^2$  function  $q$  which touches  $\varphi$  from above at  $x_0$  then

$$H_m(q) \geq F(x, q) \beta^n, \text{ at } x_0.$$

Here we use the notation  $H_m(u) = (dd^c u)^m \wedge \beta^{n-m}$  for  $u \in \mathcal{C}^2(X)$ . We also say that  $H_m(\varphi) \geq F(x, q) \beta^n$  “in the viscosity sense”.

**Definition 3.3.** A lower semicontinuous function  $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$  is a viscosity supersolution to (6) if  $\varphi \not\equiv +\infty$  and for any  $x_0 \in X$  and any  $\mathcal{C}^2$  function  $q$  which touches  $\varphi$  from below at  $x_0$  then

$$[(dd^c q)^m \wedge \beta^{n-m}]_+ \leq F(x, q) \beta^n, \text{ at } x_0.$$

Here  $[\alpha^m \wedge \beta^{n-m}]_+$  is defined to be itself if  $\alpha$  is  $m$ -positive and 0 otherwise.

**Remark 3.4.** If  $u \in \mathcal{C}^2(\Omega)$  then  $H_m(\varphi) \geq F(x, \varphi) \beta^n$  (or  $[H_m(\varphi)]_+ \leq F(x, \varphi) \beta^n$ ) holds in the viscosity sense iff it holds in the usual sense.

**Definition 3.5.** A continuous function  $\varphi : X \rightarrow \mathbb{R}$  is a viscosity solution to (1) if it is both a subsolution and a supersolution.

The notion of viscosity subsolutions is stable under taking maximum. It is also stable along monotone sequences as the following lemma shows.

**Lemma 3.6.** *Assume that  $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^+$  is a continuous function. Let  $(\varphi_j)$  be a monotone sequence of viscosity subsolutions of equation*

$$(7) \quad -(dd^c u)^m \wedge \beta^{n-m} + F(x, u)\beta^n = 0,$$

*If  $\varphi_j$  is uniformly bounded from above and  $\varphi := (\lim \varphi_j)^* \not\equiv -\infty$  then  $\varphi$  is also a viscosity subsolution of (7).*

*Proof.* The proof can be found in [CIL92]. For convenience, we reproduce it here. Observe that if  $z_j \rightarrow z$  then

$$\limsup_{j \rightarrow +\infty} \varphi_j(z_j) \leq \varphi(z).$$

Fix  $x_0 \in \Omega$  and  $q$  a  $\mathcal{C}^2$  function in a neighborhood of  $x_0$ , say  $B(x_0, r) \subset \Omega$  which touches  $\varphi$  from above at  $x_0$ . We can choose a sequence  $(x_j) \subset B = \bar{B}(x_0, r/2)$  converging to  $x_0$  and a subsequence  $(\varphi_j)$  (still denoted by  $\varphi_j$ ) such that

$$\varphi_j(x_j) \rightarrow \varphi(x_0).$$

Fix  $\epsilon > 0$ . For each  $j$ , let  $y_j$  be the maximum point of  $\varphi_j - q - \epsilon|x - x_0|^2$  on  $B$ . Then

$$(8) \quad \varphi_j(x_j) - q(x_j) - \epsilon|x_j - x_0|^2 \leq \varphi_j(y_j) - q(y_j) - \epsilon|y_j - x_0|^2.$$

We claim that  $y_j \rightarrow x_0$ . Indeed, assume that  $y_j \rightarrow y \in B$ . Letting  $j \rightarrow +\infty$  in (8) and noting that  $\limsup \varphi_j(y_j) \leq \varphi(y)$ , we get

$$0 \leq \varphi(y) - q(y) - \epsilon|y - x_0|^2.$$

Remember that  $q$  touches  $\varphi$  above in  $B$  at  $x_0$  and  $y \in B$ . Thus, the above inequality implies that  $y = x_0$ , which means  $y_j \rightarrow x_0$ . Then again by (8) we deduce that  $\varphi_j(y_j) \rightarrow \varphi(x_0)$ .

For  $j$  large enough, the function

$$q + \epsilon|x - x_0|^2 + \varphi_j(y_j) - q(y_j) - \epsilon|y_j - x_0|^2$$

touches  $\varphi_j$  from above at  $y_j$ . Thus

$$H_m(q + \epsilon|x - x_0|^2)(y_j) \geq F(y_j, \varphi_j(y_j))\beta^n.$$

It suffices now to let  $j \rightarrow +\infty$ . □

When  $F \equiv 0$ , viscosity subsolutions of (1) are exactly  $m$ -subharmonic functions.

**Lemma 3.7.** *A function  $u$  is  $m$ -subharmonic in  $\Omega$  if and only if it is a viscosity subsolution of*

$$(9) \quad (dd^c u)^m \wedge \beta^{n-m} = 0.$$

*Proof.* Assume that  $u$  is  $m$ -subharmonic in  $\Omega$  and let  $u_\epsilon$  be its standard smooth regularization. Then  $u_\epsilon$  is  $m$ -subharmonic and smooth, hence  $u_\epsilon$  is a classical subsolution of (9). Thus, it follows from Lemma 3.6 that  $u$  is a viscosity subsolution of (9).

Conversely, assume that  $u$  is a viscosity subsolution of (9). Fix  $\alpha_1, \dots, \alpha_{m-1}$   $m$ -positive  $(1,1)$ -forms with constant coefficients such that

$$\alpha_1 \wedge \dots \wedge \alpha_{m-1} \wedge \beta^{n-m}$$

is strictly positive. Let  $x_0 \in \Omega$  and  $q \in \mathcal{C}^2(V_{x_0})$  such that  $u - q$  has a local maximum at  $x_0$ . Then for any  $\epsilon > 0$ ,  $q + \epsilon|z - z_0|^2$  also touches  $u$  from above. By the definition of viscosity subsolutions, we have

$$(dd^c q + \epsilon\beta)^m \wedge \beta^{n-m} \geq 0, \forall \epsilon > 0,$$

which means that the Hessian matrix  $\frac{\partial^2 q}{\partial z_j \partial \bar{z}_k}(x_0)$  is  $m$ -positive. Hence

$$L_\alpha q := dd^c q \wedge \alpha_1 \wedge \dots \wedge \alpha_{m-1} \wedge \beta^{n-m} \geq 0,$$

holds at  $x_0$ .

This implies  $L_\alpha u \geq 0$  in the viscosity sense. In appropriate complex coordinates this constant coefficient differential operator is the Laplace operator. Hence, [Hor94] prop 3.2.10' p. 147 implies that  $u$  is  $L_\alpha$ -subharmonic hence is  $L_{\text{loc}}^1(V_{x_0})$  and satisfies  $L_\alpha u \geq 0$  in the sense of distributions. Since  $\alpha_1, \dots, \alpha_{m-1}$  was taken arbitrarily, by continuity we have

$$dd^c u \wedge \alpha_1 \wedge \dots \wedge \alpha_{m-1} \wedge \beta^{n-m} \geq 0$$

in the sense of distributions for any  $m$ -positive  $(1,1)$ -forms  $\alpha$ . Therefore,  $u$  is  $m$ -subharmonic.  $\square$

**Corollary 3.8.** *Lemma 3.6 still holds if the sequence  $\varphi_j$  is not monotone.*

*Proof.* For each  $j$ , set

$$u_j = (\sup_{k \geq j} \varphi_k)^*, \quad v_l := \max(\varphi_j, \dots, \varphi_{j+l}).$$

Since the notion of viscosity subsolution is stable under taking the maximum, we deduce that  $v_l$  is a viscosity subsolution of (7). Observe that  $u_j = (\sup_{l \geq 0} v_l)^*$  and the sequence  $(v_l)$  is monotone. It follows from what we have done before that  $u_j$  is a viscosity subsolution of (7). By Lemma 3.7 each  $\varphi_j$  is  $m$ -subharmonic. Hence,  $u_j \downarrow \varphi$  and the proof is complete.  $\square$

For real  $(1,1)$ -form  $\alpha$ , we denote by

$$S_m(\alpha) := \frac{\alpha^m \wedge \beta^{n-m}}{\beta^n}.$$

Set

$$U_m := \{\alpha \in \tilde{\Gamma}_m \text{ of constant coefficients such that } S_m(\alpha) = 1\}.$$

It is elementary to prove the following lemma:

**Lemma 3.9.** *Let  $\beta$  be a real  $m$ -positive  $(1,1)$ -form. Then the following identity holds*

$$(S_m(\beta))^{1/m} = \inf \left\{ \frac{\beta \wedge \alpha_1 \wedge \dots \wedge \alpha_{m-1} \wedge \beta^{n-m}}{\beta^n} \mid \alpha_j \in U_m, \forall j \right\}.$$

Now, we compare viscosity and potential subsolutions when the right hand side  $F(x, t)$  does not depend on  $t$ .

**Proposition 3.10.** *Let  $\varphi$  be a bounded upper semicontinuous function in  $\Omega$  and  $0 \leq f$  be a continuous function.*

(i) *If  $\varphi$  is  $m$ -subharmonic such that*

$$(10) \quad H_m(\varphi) \geq f\beta^n$$

*in the potential sense then it also holds in the viscosity sense.*

(ii) *Conversely, if (10) holds in the viscosity sense then  $\varphi$  is  $m$ -subharmonic and the inequality holds in the potential sense.*

*Proof.* We follows [EGZ11].

**Proof of (i):** Let  $x_0 \in \Omega$  and assume that  $q$  is a  $C^2$  functions which touches  $\varphi$  from above at  $x_0$ . Suppose that  $H_m(q(x_0)) < f(x_0)\beta^n$ . There exists  $\epsilon > 0$  such that  $H_m(q_\epsilon) < f\beta^n$  in a neighborhood of  $x_0$  since  $f$  is continuous, here  $q_\epsilon = q + \epsilon|z - x_0|^2$ . It follows from the proof of Lemma 3.7 that  $q_\epsilon$  is  $m$ -subharmonic in a neighborhood



of  $x_0$ , say  $B$ . Now, for  $\delta > 0$  small enough, we have  $q_\epsilon - \delta \geq \varphi$  on  $\partial B$  but it fails at  $x_0$  which contradicts the comparison principle.

**Proof of (ii):** We proceed steps by steps.

**Step 1: Assume that  $0 < f^{1/m}$  is Hölder continuous.** Let  $x_0 \in \Omega$  and assume that  $q$  is a  $C^2$  functions which touches  $\varphi$  from above at  $x_0$ . Fix  $\alpha_1, \dots, \alpha_{m-1} \in U_m$ .

We can find  $h \in C^2(\{x_0\})$  such that  $L_\alpha h = f^{1/m} \beta^n$ . As in the proof of Lemma 3.7, we can prove that  $\varphi - h$  is  $L_\alpha$ -subharmonic, which gives  $L_\alpha \varphi \geq L_\alpha h = f^{1/m} \beta^n$  in the potential sense.

Consider the standard regularization  $\varphi_\epsilon$  of  $\varphi$  by convolution with a smoothing kernel. Then

$$L_\alpha \varphi_\epsilon \geq (f^{1/m})_\epsilon \beta^n,$$

in the potential sense and hence in the usual sense. Now, use Lemma 3.9, we obtain

$$H_m(\varphi_\epsilon) \geq (f^{1/m})_\epsilon^m \beta^n.$$

Letting  $\epsilon \rightarrow 0$  and noting that the Hessian operator is continuous under decreasing sequence, we get

$$H_m(\varphi) \geq f \beta^n.$$

**Step 2: Assume that  $0 < f$  is only continuous.** Note that

$$f = \sup\{h \in C^\infty(\Omega), 0 < h \leq f\}.$$

Now, if  $H_m(\varphi) \geq f \beta^n$  in the viscosity sense then we also have  $H_m(\varphi) \geq h \beta^n$  in the viscosity sense provided that  $f \geq h$ . Thus, by step 1,

$$H_m(\varphi) \geq h \beta^n,$$

for every  $0 < h \leq f \in C^\infty(\Omega)$ . This yields

$$H_m(\varphi) \geq f \beta^n$$

in the viscosity sense.

**Step 3:  $0 \leq f$  is merely continuous.** We consider  $\varphi_\epsilon = \varphi + \epsilon|z|^2$ . Then

$$H_m(\varphi_\epsilon) \geq (f + \epsilon^m) \beta^n$$

in the viscosity sense. By Step 2 we have

$$H_m(\varphi_\epsilon) \geq (f + \epsilon^m) \beta^n$$

in the potential sense and the result follows by letting  $\epsilon$  go to 0.  $\square$

**Theorem 3.11.** *Let  $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^+$  be a continuous function which is non-decreasing in the second variable. Let  $\varphi$  be a bounded u.s.c. function in  $\Omega$ . Then the inequality*

$$(11) \quad H_m(\varphi) \geq F(x, \varphi) \beta^n$$

*holds in the viscosity sense if and only if  $\varphi$  is  $m$ -subharmonic in  $\Omega$  and (11) holds in the potential sense.*

*Proof.* Let us prove the first implication. Assume that (11) holds in the viscosity sense. Consider the sup-convolution of  $\varphi$ :

$$(12) \quad \varphi^\delta(x) := \sup\{\varphi(y) - \frac{1}{\delta^2}|x - y|^2 \mid y \in \Omega\}, \quad x \in \Omega_\delta,$$

where  $\Omega_\delta := \{x \in \Omega \mid d(x, \partial\Omega) > A\delta\}$ , and the positive constant  $A$  is chosen so that  $A^2 > \text{osc}_\Omega \varphi$ . Then  $\varphi^\delta \downarrow \varphi$  and as in [Ish89] (see also [EGZ11]) it can be shown that

$$(13) \quad H_m(\varphi^\delta) \geq F_\delta(x, \varphi^\delta) \beta^n, \quad \text{in } \Omega_\delta,$$

in the viscosity sense, where  $F_\delta(x, t) = \inf_{|y-x| \leq A\delta} F(y, t)$ .

It follows from Proposition 3.10 that (13) holds in the potential sense and the result follows by letting  $\delta$  go to 0.

Let us prove the other implication. Suppose that  $\varphi$  satisfies (11) in the potential sense. As in [EGZ11] it can be shown that

$$(14) \quad H_m(\varphi^\delta) \geq F_\delta(x, \varphi^\delta)\beta^n,$$

in the potential sense. Now, applying Proposition 3.10 to  $\varphi^\delta$  we see that (14) holds in the viscosity sense. It suffices to let  $\delta \rightarrow 0$ .  $\square$

#### 4. LOCAL COMPARISON PRINCIPLE

In this section we follow [YW10] (see also [CC95]) to prove a viscosity comparison principle for equation (6).

**Definition 4.1.** A function  $u : \Omega \rightarrow \mathbb{R}$  is called semi concave (resp. semi convex) if there exists  $K > 0$  (resp.  $K < 0$ ) such that for every  $z_0 \in \Omega$  there exists a quadratic polynomial  $P = K|z|^2 + l$ , where  $l$  is an affine function, which touches  $u$  from above (resp. below) at  $z_0$ .

**Definition 4.2.** A function  $u : \Omega \rightarrow \mathbb{R}$  is called punctually second order differentiable at  $z_0 \in \Omega$  if there exists a quadratic polynomial  $q$  such that

$$u(z) = q(z) + o(|z - z_0|^2) \text{ as } z \rightarrow z_0.$$

Note that such a  $q$  is unique if it exists. We thus define  $dd^c u(z_0), D^2 u(z_0)$  to be  $dd^c q(z_0), D^2 q(z_0)$ .

The following result is a theorem of Alexandroff-Buselman-Feller (see [EG92, Theorem 1, Section 6.4], or [Kr87, Theorem 1, Section 1.2], or [Kr87, Appendix 2]).

**Theorem 4.3.** *Every continuous semi convex (or semi concave) function is punctually second order differentiable almost everywhere.*

**Theorem 4.4** (Local comparison principle). *Let  $F$  be a continuous function which is non-decreasing in the second variable. Let  $u$  be a viscosity subsolution and  $v$  be a viscosity supersolution of*

$$H_m(\varphi) = F(x, \varphi)\beta^n.$$

*If  $u \leq v$  on  $\partial\Omega$  then  $u \leq v$  on  $\Omega$ .*

*Proof.* By considering  $u - \epsilon$ ,  $\epsilon > 0$  and then letting  $\epsilon \rightarrow 0$  noting that  $F$  is non decreasing in the second variable, we can assume that  $u < v$  near the boundary of  $\Omega$ . Assume by contradiction that there exists  $x_0 \in \Omega$  such that

$$u(x_0) - v(x_0) = a > 0.$$

Let  $u^\epsilon, v_\epsilon$  be the sup-convolution and inf-convolution (which is defined similarly as in (12)). They are semi convex and semi concave functions respectively. By Dini's Lemma  $w_\epsilon := v_\epsilon - u_\epsilon \geq 0$  near the boundary  $\partial\Omega$  for  $\epsilon > 0$  small enough. Thus, we can fix some open subset  $U \Subset \Omega$  such that  $w_\epsilon \geq 0$  on  $\Omega \setminus U$ .

Fix  $\epsilon > 0$  small enough. Denote by  $E_\epsilon$  the set of all points in  $U$  where  $w_\epsilon, u^\epsilon, v_\epsilon$  are punctually second order differentiable. Then by Theorem 4.3, the Lebesgue measure of  $U \setminus E_\epsilon$  is 0. Fix some  $r > 0$  such that  $\Omega \subset B_r \subset B_{2r}$ . Define

$$G_\epsilon(x) = \sup\{\varphi(x) \mid \varphi \text{ is convex in } B_{2r}, \varphi \leq \min(w_\epsilon, 0) \text{ in } \Omega\}.$$

Since  $w_\epsilon \geq 0$  on  $\partial U$  and  $w_\epsilon(x_0) \leq a < 0$ , using Alexandroff-Bakelman-Pucci (ABP) estimate (see also [YW10, Lemma 4.7]) we can find  $x_\epsilon \in E_\epsilon$  such that

- (i)  $w_\epsilon(x_\epsilon) = G_\epsilon(x_\epsilon) < 0$ ,
- (ii)  $G_\epsilon$  is punctually second order differentiable at  $x_\epsilon$  and  $\det_{\mathbb{R}}(D^2 G_\epsilon(x_\epsilon)) \geq \delta$ , where  $\delta > 0$  depends only on  $a, n$  and  $\text{diam}(\Omega)$ .

Since  $G_\epsilon$  is convex, we also have  $\det_{\mathbb{C}}(dd^c G_\epsilon)(x_\epsilon) \geq \delta^{1/2}$ . It follows from Gårding's inequality [Ga59] that

$$(dd^c G_\epsilon)^m \wedge \beta^{n-m}(x_\epsilon) \geq \delta_1 \beta^n,$$

where  $\delta_1$  does not depend on  $\epsilon$ . On the other hand,

$$H_m(u^\epsilon)(x_\epsilon) \geq F_\epsilon(x_\epsilon, u^\epsilon(x_\epsilon))\beta^n,$$

Moreover  $G_\epsilon + u^\epsilon$  touches  $v_\epsilon$  from below at  $x_\epsilon$ . Since  $G_\epsilon + u^\epsilon$  is  $m$ -subharmonic and punctually second order differentiable at  $x_\epsilon$  it follows that

$$H_m(G_\epsilon + u^\epsilon)(x_\epsilon) \leq F^\epsilon(x_\epsilon, v_\epsilon(x_\epsilon))\beta^n.$$

Since  $F$  is non-decreasing in the second variable and since  $w_\epsilon(x_\epsilon) < 0$ , the above inequality implies that

$$\delta_2 + F_\epsilon(x_\epsilon, u^\epsilon(x_\epsilon)) \leq F^\epsilon(x_\epsilon, u^\epsilon(x_\epsilon)),$$

where  $\delta_2 > 0$  is another constant which does not depend on  $\epsilon$ . Letting  $\epsilon \rightarrow 0$ , after a subsequence if necessary we obtain a contradiction.  $\square$

## 5. VISCOSITY SOLUTIONS ON HOMOGENEOUS COMPACT HERMITIAN MANIFOLDS

In this section we consider viscosity solutions to

$$(15) \quad (\omega + dd^c \varphi)^m \wedge \omega^{n-m} = F(x, \varphi)\omega^n,$$

where  $(X, \omega)$  satisfies (H1), (H2) and (H3).

The notion of viscosity subsolutions and supersolutions are defined similarly as in the local case. We compare viscosity and potential subsolutions in the two following theorems.

**Proposition 5.1.** *Assume that  $\omega$  is Kähler and  $\varphi$  is a continuous function on  $X$ . Then  $\varphi$  is  $(\omega, m)$ -subharmonic iff*

$$(16) \quad (\omega + dd^c \varphi)^m \wedge \omega^{n-m} \geq 0$$

*in the viscosity sense.*

*Proof.* Assume that  $\varphi$  is  $(\omega, m)$ -subharmonic and let  $\varphi_\epsilon$  be the smooth regularizing sequence of  $\varphi$  as in [Chi12]. Then  $\varphi_\epsilon$  is  $(\omega, m)$ -subharmonic in the viscosity sense.

Fix  $x_0 \in X$ ,  $\delta > 0$  and  $q$  a  $C^2$  function which touches  $\varphi$  from above at  $x_0$ . Let  $B$  be a small closed ball where the touching appears and let  $x_\epsilon$  be a maximum point of  $\varphi_\epsilon - q - \delta\rho$  in  $B$ . Here  $\rho = |z - x_0|^2$ . Then due to the uniform convergence of  $\varphi_\epsilon$  and Dini's Lemma we have  $x_\epsilon \rightarrow x_0$  as  $\epsilon \downarrow 0$ . Also, for small  $\epsilon > 0$ ,  $q + \delta\rho + \varphi_\epsilon(x_\epsilon) - q(x_\epsilon)$  touches  $\varphi_\epsilon$  from above at  $x_\epsilon$ . This implies that

$$(\omega + dd^c q + \delta dd^c \rho)^m \wedge \omega^{n-m} \geq 0$$

holds at  $x_\epsilon$  which, in turn, implies one implication by letting  $\epsilon \downarrow 0$  and  $\delta \downarrow 0$ .

Let us prove the other implication. Assume that  $\varphi$  satisfies (16) in the viscosity sense. Fix  $\alpha = \alpha_1 \wedge \dots \wedge \alpha_{m-1}$ , where  $\alpha_i$  are smooth  $(\omega, m)$ -positive closed  $(1,1)$ -forms. By Gårding's inequality we see that

$$(\omega + dd^c \varphi) \wedge \alpha \wedge \omega^{n-m} \geq 0$$

in the viscosity sense. Thanks to [HKM93, Corollary 7.20] the same arguments as in [Hor94, page 147] show that the above inequality also holds in the sense of currents. Thus,  $\varphi$  is  $(\omega, m)$ -subharmonic.  $\square$

**Theorem 5.2.** *Assume that  $\omega$  is Kähler,  $F$  is continuous on  $X \times \mathbb{R}$  and increasing in the second variable, and  $\varphi \in \mathcal{C}(X)$ . Then  $\varphi$  is  $(\omega, m)$ -subharmonic and satisfies*

$$(17) \quad (\omega + dd^c \varphi)^m \wedge \omega^{n-m} \geq F(x, \varphi)\omega^n$$

*in the potential sense if and only if the above inequality holds in the viscosity sense.*

*Proof.* Set

$$f(x) = F(x, \varphi(x)), \quad x \in X.$$

Assume that  $\varphi$  satisfies (17) in the potential sense. Let  $x_0 \in X$  and  $q \in \mathcal{C}^2(U)$  which touches  $\varphi$  from above at  $x_0$  in  $U$ , a small neighborhood of  $x_0$ . Suppose by contradiction that

$$(\omega + dd^c q)^m \wedge \omega^{n-m} < f\omega^n$$

holds at  $x_0$ . Then for  $\epsilon$  small enough we have

$$(\omega + dd^c q_\epsilon)^m \wedge \omega^{n-m} < f\omega^n$$

in a small ball  $B$  containing  $x_0$ . Here  $q_\epsilon = q + \epsilon|x - x_0|^2$  defined in a local chart near  $x_0$ . Since  $q$  touches  $\varphi$  from above at  $x_0$  in  $B$ , we can find  $\delta > 0$  small enough such that  $q_\epsilon - \delta \geq \varphi$  on  $\partial B$ . But  $q_\epsilon(x_0) - \delta < \varphi(x_0)$  which contradicts the potential comparison principle.

Now, we prove the other implication. Assume that  $\varphi$  satisfies (17) in the viscosity sense. Then from Proposition 5.1 we see that  $\varphi$  is  $(\omega, m)$ -subharmonic.

We consider two cases.

**Case 1:  $F$  does not depend on the second variable.** We denote  $f(x) = F(x, 0)$  for  $x \in X$ .

We first treat the case when  $f > 0$ . Fix  $\tilde{f}$  a smooth function such that  $0 < \tilde{f} \leq f$ . Then  $\varphi$  satisfies

$$(\omega + dd^c \varphi)^m \wedge \omega^{n-m} \geq \tilde{f}\omega^n$$

in the viscosity sense.

Fix  $\alpha = \alpha_1 \wedge \dots \wedge \alpha_{m-1}$ , where  $\alpha_i$  are smooth  $(\omega, m)$ -positive closed  $(1,1)$ -forms and set

$$\alpha_j^m \wedge \omega^{n-m} = h_j \omega^n, \quad j = 1 \dots m-1.$$

From Gårding's inequality we see that

$$(18) \quad (\omega + dd^c \varphi) \wedge \alpha \wedge \omega^{n-m} \geq h_1^{1/m} \dots h_{m-1}^{1/m} \tilde{f}^{1/m} \omega^n,$$

in the viscosity sense. As in the proof of Proposition 5.1 it also holds in the potential sense. Let  $\varphi_\epsilon$  be the smooth regularization of  $\varphi$  constructed in [Chi12]. We claim that

$$(\omega + dd^c \varphi_\epsilon) \wedge \alpha \wedge \omega^{n-m} \geq h_1^{1/m} \dots h_{m-1}^{1/m} (\tilde{f}^{1/m})_\epsilon \omega^n,$$

in the usual sense pointwise on  $X$ . Indeed, recall the definition of  $\varphi_\epsilon$ :

$$\varphi_\epsilon(x) = \int_K \mathcal{L}_g^* \varphi(x) \chi_\epsilon(g) dg,$$

where by  $\mathcal{L}_g$  we denote the left translation by  $g$ , i.e  $\mathcal{L}_g(x) = g.x$ ,  $\forall x \in X$ . We compute

$$\begin{aligned} (\omega + dd^c \varphi_\epsilon) \wedge \alpha \wedge \omega^{n-m} &= \int_K \mathcal{L}_g^* \left( (\omega + dd^c \varphi) \wedge \mathcal{L}_{g^{-1}}^* \alpha \wedge \omega^{n-m} \right) \chi_\epsilon(g) dg \\ (\text{By (18)}) &\geq \int_K \mathcal{L}_g^* \left( \mathcal{L}_{g^{-1}}^* (h_1^{1/m} \dots h_{m-1}^{1/m} \tilde{f}^{1/m} \omega^n) \right) \chi_\epsilon(g) dg \\ &= h_1^{1/m} \dots h_{m-1}^{1/m} (\tilde{f}^{1/m})_\epsilon \omega^n. \end{aligned}$$

Thus, the claim is proved. By choosing  $\alpha_j = (\omega + dd^c \varphi_\epsilon)$ ,  $j = 1, \dots, m-1$  it follows that

$$(\omega + dd^c \varphi_\epsilon)^m \wedge \omega^{n-m} \geq ((\tilde{f}^{1/m})_\epsilon)^m \omega^n.$$

By letting  $\epsilon \downarrow 0$  we get

$$(\omega + dd^c \varphi)^m \wedge \omega^{n-m} \geq \tilde{f} \omega^n,$$

in the potential sense. Since  $\tilde{f}$  was chosen arbitrarily, we deduce that

$$(\omega + dd^c \varphi)^m \wedge \omega^{n-m} \geq f \omega^n,$$

in the viscosity sense.

If  $0 \leq f$  is continuous we consider  $\varphi_t := (1-t)\varphi + t\psi$  where  $\psi$  is a smooth strictly  $(\omega, m)$ -subharmonic function and  $0 < t < 1$ . Then for each fixed  $t \in (0, 1)$ ,  $\varphi_t$  satisfies

$$(19) \quad (\omega + dd^c \varphi_t)^m \wedge \omega^{n-m} \geq f_t \omega^n,$$

in the viscosity sense with  $f_t$  continuous and strictly positive:

$$f_t = (1-t)^m f + t^m \frac{(\omega + dd^c \psi)^m \wedge \omega^{n-m}}{\omega^n}.$$

We then can apply what we have done above to infer that  $\varphi_t$  verifies (19) in the potential sense. It suffices now to let  $t \downarrow 0$ .

**Case 2:  $F$  depends on the second variable.** Since  $\varphi$  is continuous, the function  $f : X \rightarrow \mathbb{R}$ ,  $f(x) = F(x, \varphi(x))$  is continuous. We can apply case 1 to complete the proof.  $\square$

**5.1. Global Comparison Principle.** Let  $u, v$  be a subsolution and a supersolution of (15). Construct a distance  $d$  on  $K$  such that  $d^2 : K \times K \rightarrow \mathbb{R}^+$  is smooth. Consider the sup-convolution and inf-convolution as follows

$$(20) \quad u^\epsilon(x) := \sup \left\{ u(g.x) - \frac{1}{\epsilon^2} d^2(g, e) \mid g \in K \right\},$$

and

$$(21) \quad v_\epsilon(x) := \inf \left\{ v(g.x) + \frac{1}{\epsilon^2} d^2(g, e) \mid g \in K \right\}.$$

**Lemma 5.3.** *Fix  $x_0 \in X$  and consider local coordinates  $z : \Omega \rightarrow B(0, 2)$ , where  $\Omega$  is a small open neighborhood of  $x_0$  and  $B$  is the ball of radius 2 in  $\mathbb{C}^n$ . Then  $u^\epsilon, v_\epsilon$  read in this local chart as semiconvex and semiconcave functions. In particular, they are punctually second order differentiable almost everywhere in  $B(0, 1)$ .*

*Proof.* We only need to prove the result for  $u^\epsilon$  since for  $v_\epsilon$  it follows similarly. Consider a smooth section  $s : \Omega \rightarrow K$  such that  $\pi \circ s(x) = x, \forall x \in \Omega$ , where  $\pi$  is the projection of  $K$  onto  $X$ .

For simplicity we identify a point in  $\Omega$  with its image in  $B(0, 2)$ .

Put

$$\rho(x) = u^\epsilon + C|x|^2,$$

where  $C > 0$  is a big constant to be specified later.

We claim that for any  $x \in B(0, 1)$  there exists  $\delta > 0$  such that

$$\rho(x+h) + \rho(x-h) \geq 2\rho(x), \quad \forall h \in \mathbb{C}^n, |h| \leq \delta.$$

It is classical that this property implies the convexity of  $\rho$ . Let us prove the claim. Let  $x_0 \in B(0, 1)$  and  $y_0 = g_0.x_0$  be such that

$$(22) \quad u^\epsilon(x_0) = u(y_0) - \frac{1}{\epsilon^2} d^2(g_0, e).$$

By considering  $\epsilon > 0$  small enough we can assume that  $y_0 \in B(0, 3/2)$ . For  $h \in \mathbb{C}^n$  small enough such that  $x_0 + h, x_0 - h \in B(0, 1)$ , set

$$\theta(h) = g_0.s(x_0).s(x_0 + h)^{-1}.$$

Then it is easy to see that  $\theta(h).(x_0 + h) = y_0$ . By definition of  $u^\epsilon$  we thus get

$$(23) \quad u^\epsilon(x_0 + h) \geq u(y_0) - \frac{1}{\epsilon^2} d^2(\theta(h), e),$$

and

$$(24) \quad u^\epsilon(x_0 - h) \geq u(y_0) - \frac{1}{\epsilon^2} d^2(\theta(-h), e).$$

From (22), (23) and (24) we obtain

$$u^\epsilon(x_0 + h) + u^\epsilon(x_0 - h) - 2u^\epsilon(x_0) \geq -\frac{1}{\epsilon^2} \left( d^2(\theta(h), e) + d^2(\theta(-h), e) - 2d^2(\theta(0), e) \right).$$

Since  $s$  is smooth and  $K$  is compact we can choose  $C > 0$  big enough (does not depend on  $x_0$ ) such that

$$u^\epsilon(x_0 + h) + u^\epsilon(x_0 - h) - 2u^\epsilon(x_0) \geq -2C|h|^2,$$

for  $h \in \mathbb{C}^n$  small enough. This proves the claim. The last statement follows from Alexandroff-Buselman-Feller's theorem (Theorem 4.3).  $\square$

**Lemma 5.4.**  $u^\epsilon$  is a viscosity subsolution of

$$(25) \quad -(\omega + dd^c u)^m \wedge \omega^{n-m} + F_\epsilon(x, u)\omega^n = 0,$$

where

$$F_\epsilon(x, t) := \inf \left\{ F(g.x, t) \mid g \in K, d(g, e) \leq \sqrt{\text{osc}(u)\epsilon} \right\}.$$

Similarly,  $v_\epsilon$  is a viscosity supersolution of

$$(26) \quad -(\omega + dd^c u)^m \wedge \omega^{n-m} + F^\epsilon(x, u)\omega^n = 0,$$

where

$$F^\epsilon(x, t) := \sup \left\{ F(g.x, t) \mid g \in K, d(g, e) \leq \sqrt{\text{osc}(v)\epsilon} \right\}.$$

*Proof.* We only need to prove the first assertion since the second one follows similarly. Let  $q$  be a function of class  $\mathcal{C}^2$  in a neighborhood of  $x_0 \in X$  that touches  $u^\epsilon$  from above at  $x_0$ . Let  $g_0 \in K$  be such that

$$u^\epsilon(x_0) = u(g_0.x_0) - \frac{1}{\epsilon^2} d^2(g_0, e).$$

Consider the function  $Q$  defined by

$$Q(x) := q(g_0^{-1}x) + \frac{1}{\epsilon^2} d^2(g_0, e).$$

Then  $Q$  touches  $u$  from above at  $g_0.x_0$ . Since  $u$  is a subsolution of (15), we have

$$(\omega + dd^c Q)^m \wedge \omega^{n-m} \geq F(x, Q)\omega^n, \text{ at } g_0.x_0.$$

Since  $\mathcal{L}_{g_0}^* \omega = \omega$  we get

$$(\omega + dd^c q)^m \wedge \omega^{n-m} \geq F(g_0.x_0, q(x_0))\omega^n, \text{ at } x_0.$$

From the definition of  $u^\epsilon$  we know that  $u^\epsilon(x_0) = u(g_0.x_0) - \frac{1}{\epsilon^2} d^2(g_0, e) \geq u(x_0)$ . Thus  $d(g_0, e) \leq \epsilon \sqrt{\text{osc}(u)}$  and the result follows.  $\square$

Now, we prove a viscosity comparison principle on homogeneous manifolds. The fact that the metric  $\omega$  is invariant under group actions allows us to follow the proof of Theorem 4.4 in this global context.

**Theorem 5.5.** Assume that  $u, v$  are bounded viscosity subsolution and supersolution of

$$-(\omega + dd^c \varphi)^m \wedge \omega^{n-m} + F(x, \varphi)\omega^n = 0,$$

where  $0 \leq F(x, t)$  is a continuous function which is increasing in the second variable. Then we have  $u \leq v$  on  $X$ .

*Proof.* We consider the sup-convolution and inf-convolution of  $u, v$  as in (20) and (21). These functions read in local coordinates as semi-convex and semi-concave functions which are punctually second order differentiable almost every where. For each  $\epsilon > 0$  let  $x_\epsilon$  be a maximum point of  $u^\epsilon - v_\epsilon$  on  $X$ .

We first treat the case when  $u^\epsilon, v_\epsilon$  are punctually second order differentiable at  $x_\epsilon$ . In this case, by the classical maximum principle we have

$$dd^c u^\epsilon \leq dd^c v_\epsilon \text{ at } x_\epsilon.$$

The form  $(\omega + dd^c u^\epsilon)$  is  $(\omega, m)$ -positive at  $x_\epsilon$ . Thus,

$$(\omega + dd^c u^\epsilon)^m \wedge \omega^{n-m} \leq (\omega + dd^c v_\epsilon)^m \wedge \omega^{n-m} \text{ at } x_\epsilon,$$

and hence Lemma 5.4 yields

$$(27) \quad F_\epsilon(x_\epsilon, u^\epsilon) \leq F_\epsilon(x_\epsilon, v_\epsilon).$$

We can assume that  $x_\epsilon \rightarrow x_0 \in X$ . By extracting a subsequence (twice), there exists a sequence  $\epsilon_j \downarrow 0$  such that

$$F_{\epsilon_j}(x_{\epsilon_j}, u^{\epsilon_j}(x_{\epsilon_j})) \text{ and } F_{\epsilon_j}(x_{\epsilon_j}, v_{\epsilon_j}(x_{\epsilon_j}))$$

converge when  $j \rightarrow +\infty$ . We thus deduce from (27) that

$$F(x_0, \liminf_j u^{\epsilon_j}(x_{\epsilon_j})) \leq F(x_0, \limsup_j v_{\epsilon_j}(x_{\epsilon_j})).$$

Since  $F$  is increasing in the second variable the latter implies that

$$(28) \quad \liminf u^{\epsilon_j}(x_{\epsilon_j}) \leq \limsup v_{\epsilon_j}(x_{\epsilon_j}).$$

Since  $u^\epsilon \downarrow u$  and  $v_\epsilon \uparrow v$  we have

$$\sup_X (u - v) \leq \sup_X (u^{\epsilon_j} - v_{\epsilon_j}) = u^{\epsilon_j}(x_{\epsilon_j}) - v_{\epsilon_j}(x_{\epsilon_j}).$$

Then (28) implies that  $\sup_X (u - v) \leq 0$ .

Now, if  $u^\epsilon, v_\epsilon$  are not punctually second order differentiable at  $x_\epsilon$  for fixed  $\epsilon$ , we proceed as in [EGZ11] to prove that (27) still holds. Consider a local holomorphic chart centered at  $x_\epsilon$ . For simplicity we identify a point near  $x_\epsilon$  with its image in  $\mathbb{C}^n$ . For each  $k \in \mathbb{N}^*$ , the semiconvex function  $u^\epsilon - v_\epsilon - \frac{1}{2k}\|x\|^2$  attains its strict maximum at  $x_\epsilon$ . By Jensen's lemma ([Jen88]; see also [CIL92, Lemma A.3, page 60]), there exist sequences  $p_k, \xi_k \in B(0, 1/k)$  converging to 0 such that the functions  $u^\epsilon, v_\epsilon$  are punctually second order differentiable at  $x_k$  and the function

$$u^\epsilon - v_\epsilon - \frac{1}{2k}\|x\|^2 - \langle p_k, x \rangle$$

attains its local maximum at  $x_k$ . We thus get

$$dd^c u^\epsilon \leq dd^c v_\epsilon + O(1/k)\omega \text{ at } x_k.$$

Since  $v_\epsilon$  is semi-concave, and  $u^\epsilon$  is  $(\omega, m)$ -subharmonic we get

$$(\omega + dd^c u^\epsilon)^m \wedge \omega^{n-m} \leq (\omega + dd^c v_\epsilon)^m \wedge \omega^{n-m} + O(1/k)\omega^n \text{ at } x_k.$$

This together with (25) and (26) yield

$$F_\epsilon(y_k, u^\epsilon(y_k)) \leq F_\epsilon(y_k, v_\epsilon(y_k)) + O(1/k).$$

Now, let  $k \rightarrow +\infty$  we obtain (28) which completes the proof.  $\square$

## 6. PROOF OF THE MAIN RESULTS

**6.1. Proof of Theorem A.** Let  $\mathcal{F}$  denote the family of all subsolutions  $w$  of (6) such that  $u \leq w \leq v$ . It is not empty thanks to the local comparison principle. We set

$$\varphi := \sup\{w : w \in \mathcal{F}\}.$$

By Choquet's lemma  $\varphi^* = (\limsup w_j)^*$  where  $w_j$  is a sequence in  $\mathcal{F}$ . It follows from Lemma 3.6 that  $\varphi^*$  is a subsolution of (6).

We claim that  $\varphi_*$  is a supersolution of (6). Indeed, assume that  $\varphi_*$  is not a supersolution of (6). Then there exist  $x_0 \in \Omega$  and  $q \in \mathcal{C}^2(\{x_0\})$  such that  $q$  touches  $\varphi_*$  from below at  $x_0$  but

$$H_m(q)(x_0) > F(x_0, q(x_0))\beta^n.$$

By the continuity of  $F$ , we can find  $r > 0$  small enough such that  $q \leq \varphi_*$  in  $B(x_0, r)$  and

$$H_m(q)(x) > F(x, q(x))\beta^n, \quad \forall x \in B = B(x_0, r).$$

We then choose  $0 < \epsilon$  small enough and  $0 < \delta \ll \epsilon$  so that the function  $Q = Q_{\epsilon, \delta} := q + \delta - \epsilon|x - x_0|^2$  satisfies

$$H_m(Q)(x) > F(x, Q(x))\beta^n, \quad \forall x \in B.$$

Define  $\phi$  to be  $\varphi$  out side  $B$  and  $\phi = \max(\varphi, Q)$  in  $B$ . Since  $Q < \varphi$  near  $\partial B$ , we see that  $\phi$  is upper semi continuous and it is a subsolution of (6) in  $\Omega$ .

Let  $(x_j)$  be a sequence in  $B$  converging to  $x_0$  such that  $\varphi(x_j) \rightarrow \varphi_*(x_0)$ . Then  $Q(x_j) - \varphi(x_j) \rightarrow Q(x_0) - \varphi_*(x_0) = \delta > 0$ . Thus  $\phi \not\equiv \varphi$ , which contradicts the maximality of  $\varphi$ .

From the above steps we know that  $\varphi_*$  is a super solution and  $\varphi^*$  is a subsolution. We also have  $g = u_* \leq \varphi_* \leq \varphi^* \leq v^* = g$  on  $\partial\Omega$ . Thus by the viscosity comparison principle  $\varphi = \varphi_* = \varphi^*$  is a continuous viscosity solution of (1) with boundary value  $g$ .

It remains to prove that  $\varphi$  is also a potential solution of (1). From Theorem 3.11 we know that

$$H_m(\varphi) \geq F(x, \varphi)\beta^n$$

in the potential sense. Let  $B = B(x_0, r) \subset \Omega$  is a small ball in  $\Omega$ . Thanks to Dinew and Kolodziej [DK11, Theorem 2.10] we can solve the Dirichlet problem to find  $\psi \in \mathcal{P}_m(B) \cap \mathcal{C}(\bar{B})$  with boundary value  $\varphi$  such that

$$H_m(\psi) = F(x, \varphi)\beta^n, \quad \text{in } B.$$

By the potential comparison principle we have  $\varphi \leq \psi$  in  $\bar{B}$ . Define  $\tilde{\psi}$  to be  $\psi$  in  $B$  and  $\varphi$  outside. Set  $G(x) = F(x, \varphi(x)), x \in \Omega$ . It is easy to see that  $\tilde{\psi}$  is a viscosity solution of

$$-(dd^c u)^m \wedge \beta^{n-m} + G\beta^n = 0.$$

By the viscosity comparison principle we deduce that  $\tilde{\psi} \leq \varphi$  in  $\Omega$  which implies that  $\varphi = \psi$  in  $B$ . The proof is thus complete.

**6.2. Proof of Theorem B.** Let  $\varphi$  be the unique viscosity solution obtained from Theorem A. Since  $u, v$  are  $\gamma$ -Hölder continuous in  $\bar{\Omega}$  and  $F$  satisfies (4), we can find a constant  $C > 0$  such that

$$\sup_{x, y \in \bar{\Omega}} \left( |u(x) - u(y)| + |v(x) - v(y)| \right) \leq C|x - y|^\gamma,$$

and

$$\sup_{|t| \leq M} \sup_{x, y \in \bar{\Omega}} |F^{1/m}(x, t) - F^{1/m}(y, t)| \leq C|x - y|^\gamma,$$

where  $M > 0$  is such that  $|\varphi| \leq M$ , on  $\bar{\Omega}$ .



Fix  $R > 0$  such that  $\Omega \subset B(0, R)$ . Define  $\psi : \bar{\Omega} \rightarrow \mathbb{R}$  by

$$\psi(x) := \sup_{y \in \bar{\Omega}} \left\{ \varphi(y) + C|x - y|^\gamma(|x|^2 - R^2 - 1) \right\}.$$

**Step 1: Prove that  $\psi$  is  $\gamma$ -Hölder continuous.** Fix  $x_1, x_2 \in \bar{\Omega}$ , and  $y_1, y_2$  corresponding maximum points in  $\bar{\Omega}$  as in the definition of  $\psi$ . We obtain

$$\begin{aligned} \psi(x_1) - \psi(x_2) &\geq C|x_1 - y_2|^\gamma(|x_1|^2 - R^2 - 1) - C|x_2 - y_2|^\gamma(|x_2|^2 - R^2 - 1) \\ &= C(|x_1|^2 - R^2 - 1)(|x_1 - y_2|^\gamma - |x_2 - y_2|^\gamma) + C|x_2 - y_2|^\gamma(|x_1|^2 - |x_2|^2) \\ &\geq C(|x_1|^2 - R^2 - 1)|x_1 - x_2|^\gamma + C|x_2 - y_2|^\gamma(|x_1|^2 - |x_2|^2) \geq -C'|x_1 - x_2|^\gamma, \end{aligned}$$

where  $C' > 0$  depends only on  $C, R$ . Similarly, we have

$$\psi(x_1) - \psi(x_2) \leq C'|x_1 - x_2|^\gamma.$$

The above inequalities show that  $\psi$  is  $\gamma$ -Hölder continuous in  $\bar{\Omega}$ .

**Step 2: Prove that  $\psi$  is a subsolution of (6).** Let  $x_0 \in \Omega$  and  $q \in \mathcal{C}^2(\{x_0\})$  which touches  $\psi$  from above at  $x_0$ . Let  $y_0 \in \bar{\Omega}$  be such that

$$\psi(x_0) = \varphi(y_0) + C|x_0 - y_0|^\gamma(|x_0|^2 - R^2 - 1).$$

If  $y_0 \in \partial\Omega$  then  $\varphi(y_0) = u(y_0)$ , hence

$$\begin{aligned} 0 &\geq C|x_0 - y_0|^\gamma(|x_0|^2 - R^2) = \psi(x_0) - \varphi(y_0) + C|x_0 - y_0|^\gamma \\ &\geq u(x_0) - u(y_0) + C|x_0 - y_0|^\gamma \geq 0. \end{aligned}$$

We thus get  $\varphi(x_0) = \psi(x_0)$  and the result follows since  $\varphi$  is a subsolution. Let us treat the case  $y_0 \in \Omega$ . The function  $Q$ , defined around  $y_0$  by

$$Q(x) := q(x + x_0 - y_0) - C|x_0 - y_0|^\gamma(|x + x_0 - y_0|^2 - R^2 - 1),$$

touches  $\varphi$  from above at  $y_0$ . Since  $\varphi$  is a subsolution of (6), we have

$$\tilde{S}_m^{1/m}(dd^c Q(y_0)) \geq F^{1/m}(y_0, Q(y_0)).$$

By the concavity of  $\tilde{S}_m^{1/m}$  we get

$$\begin{aligned} \tilde{S}_m^{1/m}(dd^c q(x_0)) &\geq F^{1/m}(y_0, Q(y_0)) + C|x_0 - y_0|^\gamma \\ &= F^{1/m}(y_0, \varphi(x_0)) + C|x_0 - y_0|^\gamma \\ &\geq F^{1/m}(x_0, \varphi(x_0)), \end{aligned}$$

which implies that  $\psi$  is a subsolution of (6).

It is easy to see that  $\varphi \leq \psi$  and for any  $x \in \partial\Omega, y \in \bar{\Omega}$ , we have

$$\varphi(y) - C|x - y|^\gamma \leq v(y) - C|x - y|^\gamma \leq v(x) = g(x).$$

This implies  $\psi = g$  on  $\partial\Omega$ . Hence, since  $\varphi$  is maximal we obtain  $\varphi = \psi$  which, in turn, shows that  $\varphi$  is  $\gamma$ -Hölder continuous.

**6.3. Proof of Corollary C.** Let  $h$  be the harmonic function with boundary value  $g$ ; it is a continuous supersolution of (6). It follows from [BT76] that there exists a continuous psh function  $u$  with boundary value  $g$ . Then for  $A \gg 1$ , the function  $u + A\rho$ , where  $\rho$  is a defining function of  $\Omega$ , is a subsolution. Thus, by Theorem A there exists a continuous viscosity solution.

Now, assume that  $g$  is  $(2\gamma)$ -Hölder continuous in  $\bar{\Omega}$ . Then we can choose  $u$  to be  $\gamma$ -Hölder continuous in  $\bar{\Omega}$  thanks to [BT76]. The same thing holds for  $h$ . It suffices to apply Theorem B. The proof is thus complete.

**Remark 6.1.** In Corollary C it is natural to consider a strongly  $m$ -pseudoconvex domain (i.e. the defining function is strongly  $m$ -subharmonic). The existence of continuous subsolution and supersolution is obvious which yields the existence of viscosity solution. However, the Hölder continuity is delicate.

**6.4. Proof of Theorem D.** It follows from (5) that  $t_0$  is a subsolution and  $t_1$  is a supersolution of (15). The global comparison principle (Theorem 5.5) allows us to repeat the proof of Theorem A to prove Theorem D.

In the following, we give an example of compact hermitian homogeneous manifold satisfying our conditions (H1), (H2), (H3) which is not Kähler. It is communicated to us by Karl Oeljeklaus to whom we are indebted.

**Example 6.2.** Consider  $G = SL(3, \mathbb{C})$ ,  $K = SU(3, \mathbb{C})$  and

$$H = \left\{ \begin{pmatrix} e^w & z_1 & z_2 \\ 0 & e^{iw} & z_3 \\ 0 & 0 & e^{-w-iw} \end{pmatrix} \middle/ w, z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

Then  $G$  is a connected complex Lie group and  $H$  is a closed complex subgroup. The manifold  $X = G/H$  is Hermitian. It is clear that  $K$  acts freely and transitively on  $X$ . Taking any Hermitian metric and averaging it over the Haar measure of  $K$  we obtain a Hermitian metric  $\omega$  verifying (H3). Now we prove that  $X$  is not Kähler. Since  $K$  acts freely on  $X$  we see that  $X$  is simply connected.

Consider

$$I = \left\{ \begin{pmatrix} \lambda_1 & z_1 & z_2 \\ 0 & \lambda_2 & z_3 \\ 0 & 0 & (\lambda_1 \lambda_2)^{-1} \end{pmatrix} \middle/ z_1, z_2, z_3 \in \mathbb{C}; \lambda_1, \lambda_2 \in \mathbb{C}^* \right\}.$$

Then  $H$  is a closed subgroup of  $I$  and  $Y = G/I$  is a rational-projective manifold. If  $X$  admits a Kähler metric then it follows from [BR62] (see also [BN90]) that the Tits fibration

$$\pi : G/H \rightarrow G/I$$

is trivial and its fiber  $I/H$  is a complex compact torus. This implies that  $\pi_1(X)$  is non trivial which is impossible since  $X$  is simply connected. Thus  $X$  does not admit any Kähler metric.

## REFERENCES

- [BN90] W. Barth, R. Narasimhan (edior), *Several complex variables. VI. Complex manifolds*, Encyclopaedia of Mathematical Sciences, 69. Springer-Verlag, Berlin, 1990. viii+310.
- [BT76] E. Bedford, B. A. Taylor, *The Dirichlet problem for a complex Monge-Ampère equation*, Invent. Math. 37 (1976), no. 1, 1-44.
- [BT82] E. Bedford, B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), no. 1-2, 1-40.
- [Bl05] Z. Blocki, *Weak solutions to the complex Hessian equation*, Ann. Inst. Fourier (Grenoble) 55 (2005), no. 5, 1735-1756.
- [BR62] A. Borel, R. Remmert, *Über kompakte homogene Kählersche Mannigfaltigkeiten*, (German) Math. Ann. 145 1961/1962 429-439.
- [CC95] L. A. Caffarelli, X. Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, Vol 43, (1995).
- [CNS85] L. Caffarelli, L. Nirenberg, J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian*, Acta Math. 155 (1985), 261-301.
- [Chi12] L. H. Chinh, *Solutions to degenerate complex Hessian equations*, arXiv:1202.2436v3.
- [CIL92] M. G. Crandall, H. Ishii, P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 1-67.
- [CW01] K.-S. Chou and X.-J. Wang, *Variational theory for Hessian equations*, Comm. Pure Appl. Math., 54 (2001), 1029-1064.
- [Di09] S. Dinew, *An inequality for mixed Monge-Ampère measures*, Math. Zeit. 262 (2009), 1-15.

- [DK11] S. Dinew, S. Kolodziej, *A priori estimates for complex Hessian equations*, arXiv:1112.3063v1.
- [DK12] S. Dinew, S. Kolodziej, *Liouville and Calabi-Yau type theorems for complex Hessian equations*, arXiv:1203.3995v1.
- [EG92] C. L. Evans, R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992. viii+268 pp.
- [EGZ11] P. Eyssidieux, V. Guedj, A. Zeriahi, *Viscosity solutions to degenerate complex Monge-Ampère equations*, Comm. Pure Appl. Math. 64 (2011), no. 8, 1059-1094.
- [Ga59] L. Gårding, *An inequality for Hyperbolic Polynomials*, Journal of Mathematics and Mechanics, Vol. 8, No. 6 (1959).
- [GL96] B. Guan, Y. Y. Li *Monge-Ampère equations on Riemannian manifolds*, J. Differential Equations 132 (1996), no. 1, 126-139.
- [G99] V. Guedj, *Approximation of currents on complex manifolds*, Math. Ann. 313 (1999), no. 3, 437-474.
- [GKZ08] V. Guedj, S. Kolodziej, A. Zeriahi, *Hölder continuous solutions to Monge-Ampère equations*, Bull. Lond. Math. Soc. 40 (2008), no. 6, 1070-1080.
- [GZ05] V. Guedj, A. Zeriahi, *Intrinsic capacities on compact Kähler manifolds*, J. Geom. Anal. 15 (2005), no. 4, 607-639.
- [GZ07] V. Guedj, A. Zeriahi, *The weighted Monge-Ampère energy of quasip plurisubharmonic functions*, J. Funct. Anal. 250 (2007), no. 2, 442-482.
- [HKM93] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. vi+363 pp.
- [Hor94] L. Hörmander, *Notions of convexity*, Progress in Math., Birkhäuser (1994).
- [Hou09] Z. Hou, *Complex Hessian equation on Kähler manifold*, Int. Math. Res. Not. IMRN 2009, no. 16, 3098-3111.
- [HMW10] Z. Hou, X. Ma, D. Wu, *A second order estimate for complex Hessian equations on a compact Kähler manifold*, Math. Res. Lett. 17 (2010), no. 3, 547-561.
- [Hu94] A. Huckleberry, *Subvarieties of homogeneous and almost homogeneous manifolds*, Contributions to complex analysis and analytic geometry, 189-232, Aspects Math., E26, Friedr. Vieweg, Braunschweig, 1994.
- [Ish89] H. Ishii, *On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDEs*, Comm. Pure Appl. Math. 42 (1989), no. 1, 15-45.
- [ITW04] N. Ivchikina, N.S. Trudinger, X.-J. Wang, *The Dirichlet problem for degenerate Hessian equations*, Comm. Partial Diff. Equations 29 (2004), 219-235.
- [Jb10] A. Jbilou, *Equations hessiennes complexes sur des variétés kählériennes compactes*, C. R. Math. Acad. Sci. Paris 348 (2010), no. 1-2, 41-46.
- [Jen88] R. Jensen, *The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations*, Arch. Rational Mech. Anal. 101 (1988), no. 1, 1-27.
- [Kl91] M. Klimek, *Pluripotential theory*, London Mathematical Society Monographs. New Series, 6. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991.
- [Kok10] V. N. Kokarev, *Mixed volume forms and a complex equation of Monge-Ampère type on Kähler manifolds of positive curvature*, Izv. RAN. Ser. Mat. 74:3 (2010), 65-78.
- [Kol98] S. Kolodziej, *The complex Monge-Ampère equation*, Acta Math. 180 (1998) 69-117.
- [Kol02] S. Kolodziej, *Equicontinuity of families of plurisubharmonic functions with bounds on their Monge-Ampère masses*, Math. Z. (240) (2002), 835-847.
- [Kol03] S. Kolodziej, *The Monge-Ampère equation on compact Kähler manifolds* Indiana Univ. Math. J. 52 (2003) 667-686.
- [Kol05] S. Kolodziej *The complex Monge-Ampère equation and pluripotential theory*, Memoirs Amer. Math. Soc. 178 (2005) 64p.
- [Kr87] N. V. Krylov, *Nonlinear elliptic and parabolic equations of the second order*, Translated from the Russian by P. L. Buzytsky, Mathematics and its Applications (Soviet Series), 7. D. Reidel Publishing Co., Dordrecht, 1987. xiv+462 pp.
- [Kry95] N. V. Krylov, *On the general notion of fully nonlinear second-order elliptic equations*, Trans. Amer. Math. Soc. 347 (1995), 857-895.
- [Lab02] D. Labutin, *Potential estimates for a class of fully nonlinear elliptic equations*, Duke Math. J. 111 (2002), 1-49.
- [Li04] S. Y. Li, *On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian*, Asian J. Math. 8 (2004), no. 1, 87-106.
- [Lio83] P. L. Lions, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. II. Viscosity solutions and uniqueness*, Comm. Partial Differential Equations 8 (1983), no. 11, 1229-1276.

- [Lit63] W. Littman, *Generalized subharmonic functions: Monotonic approximations and an improved maximum principle*, Ann. Scuola Norm. Sup. Pisa (3) 17 1963 207-222.
- [SA12] A. S. Sadullaev, B. I. Abdullaev *Capacities and Hessians in a class of  $m$ -subharmonic functions*, preprint arXiv:1201.6531v2 .
- [Tr95] N. S. Trudinger, *On the Dirichlet problem for Hessian equations*, Acta Math. 175 (1995), 151-164.
- [TW99] N. S. Trudinger, X.-J. Wang, *Hessian measures II*, Ann. of Math. 150 (1999), 579-604.
- [Ur01] J. Urbas, *An interior second derivative bound for solutions of Hessian equations*, Calc. Var. PDE (12) (2001), 417-431.
- [Y78] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, Comm. Pure Appl. Math. 31 (1978), no. 3, 339-411.
- [W09] X.-J. Wang, *The  $k$ -Hessian equation*, Lect. Not. Math. 1977 (2009).
- [HW54] H. -C. Wang, *Complex paralisable manifolds*, Proc. Amer. Math. Soc. 5 (1954), 771-776.
- [YW10] Y. Wang, *A Viscosity Approach to the Dirichlet Problem for Complex Monge-Ampère Equations*, arXiv:1010.1292v2.

LU Hoang Chinh  
 Université Paul Sabatier  
 Institut de Mathématiques de Toulouse  
 118 Route de Narbonne  
 31062 Toulouse  
 lu@math.univ-toulouse.fr.